3.8 Properties of Fourier Representation

1. Signals that are periodic in time have discrete frequency-domain representations, while nonperiodic time signals have continuous frequency-domain representations: Table 3.3.

<table>
<thead>
<tr>
<th>Time-Domain Property</th>
<th>Frequency-Domain Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous</td>
<td>Nonperiodic</td>
</tr>
<tr>
<td>Discrete</td>
<td>Periodic</td>
</tr>
<tr>
<td>Periodic</td>
<td>Discrete</td>
</tr>
<tr>
<td>Nonperiodic</td>
<td>Continuous</td>
</tr>
</tbody>
</table>

3.9 Linearity and Symmetry Properties

1. Linearity property for four Fourier representations:

\[
\begin{align*}
z(t) &= ax(t) + by(t) \quad \xrightarrow{FT} \quad Z(j\omega) = aX(j\omega) + bY(j\omega) \\
Z[n] &= ax[n] + by[n] \quad \xrightarrow{DTFT} \quad Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})
\end{align*}
\]
Example 3.30 Linearity in The FS

Suppose \( z(t) \) is the periodic signal depicted in Fig. 3.49(a). Use the linearity and the results of Example 3.13 to determine the FS coefficients \( Z[k] \).

<Sol.>

1. Signal \( z(t) \):

\[
 z(t) = \frac{3}{2} x(t) + \frac{1}{2} y(t)
\]

where \( x(t) \) and \( y(t) \):

Fig. 3.49(b) and (c).

2. \( X[k] \) and \( Y[k] \):
Fourier Representations of Signals & LTI Systems

From Example 3.13.

3. FS of $z(t)$:

$$z(t) \xleftarrow{FS:2\pi} Z[k] = \left(\frac{3}{2k\pi}\right)\sin\left(k\pi/2\right) + \left(\frac{1}{2k\pi}\right)\sin\left(k\pi/2\right)$$

3.9.1 Symmetry Properties: Real and Imaginary Signals

1. FT of $x(t)$:

$$X^*(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]^* = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt$$

Equation (3.37)

2. Since $x(t)$ is real valued, $x(t) = x^*(t)$. Eq. (3.37) becomes

$$X^*(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt$$

Equation (3.38)

$X(j\omega)$ is complex-conjugate symmetric

$$X^*(j\omega) = X(-j\omega)$$

Re$\{X(j\omega)\} = $Re$\{X(-j\omega)\}$ and Im$\{X(j\omega)\} = -$Im$\{X(-j\omega)\}$
The conjugate symmetry property for DTFS:

\[ X^*[k] = X[N - k] \]

Because the DTFS coefficients are \( N \) periodic, and thus

\[ X[-k] = X[N - k] \]

(3.39)

\( x(t) \) is purely imaginary:
1. \( x^*(t) = -x(t) \).
2. Eq.(3.37) becomes

\[ X^*(j\omega) = -\int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \]

\[ X^*(j\omega) = -X(-j\omega) \]

\[ \text{Re}\{X(j\omega)\} = -\text{Re}\{X(-j\omega)\} \quad \text{and} \quad \text{Im}\{X(j\omega)\} = \text{Im}\{X(-j\omega)\} \]

Table 3.4  

<table>
<thead>
<tr>
<th>Representation</th>
<th>Real-Valued Time Signals</th>
<th>Imaginary-Valued Time Signals</th>
</tr>
</thead>
<tbody>
<tr>
<td>( FT )</td>
<td>( X^*(j\omega) = X(-j\omega) )</td>
<td>( X^*(j\omega) = -X(-j\omega) )</td>
</tr>
<tr>
<td>( FS )</td>
<td>( X^*[k] = X[-k] )</td>
<td>( X^*[k] = -X[-k] )</td>
</tr>
<tr>
<td>( DTFT )</td>
<td>( X^*(e^{j\Omega}) = X(e^{-j\Omega}) )</td>
<td>( X^*(e^{j\Omega}) = -X(e^{-j\Omega}) )</td>
</tr>
<tr>
<td>( DTFS )</td>
<td>( X^*[k] = X[-k] )</td>
<td>( X^*[k] = -X[-k] )</td>
</tr>
</tbody>
</table>
Complex-conjugate symmetry in FT:

A simple characterization of the output of an LTI system with a real-valued impulse response when the input is a real-valued sinusoid.

Continuous-time case:

1. Input signal of LTI system:

\[ x(t) = A \cos(\omega t - \phi) \]

2. Real-valued impulse response of LTI system is denoted by \( h(t) \).

3. Output signal of LTI system:

\[ y(t) = \left| H(j\omega) \right| \left( \frac{A}{2} \right) e^{j(\omega t - \phi + \arg\{H(j\omega)\})} + \left| H(-j\omega) \right| \left( \frac{A}{2} \right) e^{-j(\omega t - \phi - \arg\{H(j\omega)\})} \]

Exploiting the symmetry conditions:

\[ H(j\omega) = H(-j\omega) \]

\[ \arg\{H(j\omega)\} = -\arg\{H(-j\omega)\} \]

\[ y(t) = \left| H(j\omega) \right| A \cos(\omega t - \phi + \arg\{H(j\omega)\}) \]
The LTI system modifies the amplitude of the input sinusoid by $|H(j\omega)|$ and the phase by $\arg\{H(j\omega)\}$.

A sinusoidal input to an LTI system results in a sinusoidal output of the same frequency, with the amplitude and phase modified by the system's frequency response.

**Figure 3.50 (p. 257)**

A sinusoidal input to an LTI system results in a sinusoidal output of the same frequency, with the amplitude and phase modified by the system’s frequency response.
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- Discrete-time case:
  1. Input signal of LTI system:
     \[ x[n] = A\cos(\Omega n - \phi) \]
  2. Real-valued impulse response of LTI system is denoted by \( h[n] \).
  3. Output signal of LTI system:
     \[
     y[n] = |H(e^{j\Omega})|A\cos\left(\Omega n - \phi + \arg\{H(e^{j\Omega})}\right)
     \]

- The LTI system modifies the amplitude of the input sinusoid by \(|H(e^{j\Omega})|\) and the phase by \(\arg\{H(e^{j\Omega})\}\).

3.9.2 Symmetry Properties: Even and Odd Signals

1. \(x(t)\) is real valued and has even symmetry.
   \[ x^*(t) = x(t) \text{ and } x(-t) = x(t) \]

2. Eq.(3.37) becomes
   \[
   X^*(j\omega) = \int_{-\infty}^{\infty} x(-t) e^{-j\omega(-t)} dt
   \]
   Change of variable \(\tau = -t\)
   \[
   X^*(j\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau = X(j\omega)
   \]
3. Conclusion:

1) The imaginary part of \( X(j\omega) = 0 \):

\[
X^*(j\omega) = X(j\omega)
\]

If \( x(t) \) is real and even, then \( X(j\omega) \) is real.

2) If \( x(t) \) is real and odd, then \( X^*(j\omega) = -X(j\omega) \) and \( X(j\omega) \) is imaginary.

3.10 Convolution Properties

The convolution property is a consequence of complex sinusoids being eigenfunctions of LTI system.

3.10.1 Convolution of Nonperiodic Signals — Continuous-time case

1. Convolution of two nonperiodic continuous-time signals \( x(t) \) and \( h(t) \):

\[
y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau
\]

2. FT of \( x(t - \tau) \):

\[
x(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t-\tau)} d\omega
\]
3. Substituting this expression into the convolution integral yields

\[ y(t) = \int_{-\infty}^{\infty} h(\tau) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} e^{-j\omega \tau} d\omega \right] d\tau \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} h(\tau) e^{-j\omega \tau} d\tau \right] X(j\omega) e^{j\omega t} d\omega \]

The inner integral over \( \tau \) as the FT of \( h(\tau) \), or \( H(j\omega) \). Hence, \( y(t) \) may be written as

\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) X(j\omega) e^{j\omega t} d\omega \]

and we identify \( H(j\omega)X(j\omega) \) as the FT of \( y(t) \).

\[ y(t) = h(t) * x(t) \quad \xrightarrow{FT} \quad Y(j\omega) = X(j\omega)H(j\omega) \quad (3.40) \]

**Example 3.31 Solving A Convolution Problem in The Frequency Domain**

Let \( x(t) = (1/(\pi t))\sin(\pi t) \) be the input to a system with impulse response \( h(t) = (1/(\pi t))\sin(2\pi t) \). Find the output \( y(t) = x(t) * h(t) \).
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1. Solving problem:

Time domain → Extremely Difficult
Frequency domain → Quite Simple

2. From Example 3.26, we have

\[ x(t) = \frac{1}{\pi t} \sin(\pi t) \]

\[ y(t) = x(t) * h(t) \]

\[ h(t) \]

Convolution property

Example 3.32 Finding Inverse FT’s by Means of The Convolution Property

Use the convolution property to find \( x(t) \), where

\[ x(t) \leftrightarrow FT \rightarrow X(j\omega) = \frac{4}{\omega^2} \sin^2(\omega) \]
1. Write \( X(j\omega) = Z(j\omega) Z(j\omega) \), where

\[
Z(j\omega) = \frac{2}{\omega} \sin(\omega)
\]

2. Convolution property:

\[
z(t) * z(t) \xrightarrow{FT} Z(j\omega) Z(j\omega)
\]

\[
x(t) = z(t) * z(t)
\]

3. Using the result of Example 3.25:

\[
z(t) = \begin{cases} 
1, & |t| < 1 \\
0, & |t| > 1 
\end{cases} \xrightarrow{FT} Z(j\omega)
\]

4. Performing the convolution with itself gives the triangular waveform depicted in Fig. 3.52 (b) as the solution for \( x(t) \).
Convolution of Nonperiodic Signals — Discrete-time case

If
\[ x[n] \xleftarrow{\text{DTFT}} X(e^{j\Omega}) \]
and
\[ h[n] \xleftarrow{\text{DTFT}} H(e^{j\Omega}) \]

\[ y[n] = x[n] \ast h[n] \xleftarrow{\text{DTFT}} Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) \]  

(3.41)

3.10.2 Filtering

1. Multiplication in frequency domain \(\leftrightarrow\) Filtering.

2. The terms “filtering” implies that some frequency components of the input are eliminated while others are passed by the system unchanged.

3. System Types of filtering:
   1) Low-pass filter
   2) High-pass filter
   3) Band-pass filter

4. Realistic filter:
   1) Gradual transition band
   2) Nonzero gain of stop band

5. Magnitude response of filter:
   \[ 20 \log |H(j\omega)| \text{ or } 20 \log |H(e^{j\Omega})| \text{ [dB]} \]
Figure 3.53 (p. 263)
Frequency response of ideal continuous- (left panel) and discrete-time (right panel) filters. (a) Low-pass characteristic. (b) High-pass characteristic. (c) Band-pass characteristic.
The convolution property implies that the frequency response of a system may be expressed as the ratio of the FT or DTFT of the output to the input.

For CT system:

\[ H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \quad (3.42) \]

For DT system:

\[ H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} \quad (3.43) \]

Example 3.34 \textit{Identifying a System, Given Its Input and Output}

The output of an LTI system in response to an input \( x(t) = e^{-2t} u(t) \) is \( y(t) = e^{-t} u(t) \). Find the frequency response and the impulse response of this system.

\textbf{<Sol.>}

1. FT of \( x(t) \) and \( y(t) \):

\[ X(j\omega) = \frac{1}{j\omega + 2} \quad \text{and} \quad Y(j\omega) = \frac{1}{j\omega + 1} \]

2. Frequency response:

\[ H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \quad \text{and} \quad H(j\omega) = \frac{j\omega + 2}{j\omega + 1} \]
3. Impulse response:

\[
H(j\omega) = \left(\frac{j\omega + 1}{j\omega + 1}\right) + \frac{1}{j\omega + 1} = 1 + \frac{1}{j\omega + 1}
\]

\[
h(t) = \delta(t) + e^{-t} u(t)
\]

Recover the input of the system from the output:

**CT case**

\[
X(j\omega) = H^{\text{inv}}(j\omega)Y(j\omega)
\]

where

\[
H^{\text{inv}}(j\omega) = 1 / H(j\omega)
\]

**DT case**

\[
X(e^{j\Omega}) = H^{\text{inv}}(e^{j\Omega})Y(e^{j\Omega})
\]

where

\[
H^{\text{inv}}(e^{j\Omega}) = 1 / H(e^{j\Omega})
\]

**Example 3.35 Multipath Communication Channel: Equalization**

Consider again the problem addressed in Example 2.13. In this problem, a distorted received signal \(y[n]\) is expressed in terms of a transmitted signal \(x[n]\) as

\[
y[n] = x[n] + ax[n-1], \quad |a| < 1
\]

Use the convolution property to find the impulse response of an inverse system that will recover \(x[n]\) from \(y[n]\).
1. In Example 2.13, we have \( y[n] = x[n] \ast h[n] \), where the impulse response is

\[
h[n] = \begin{cases} 
1, & n = 0 \\
a, & n = 1 \\
0, & \text{otherwise}
\end{cases}
\]

2. The impulse response of an inverse system, \( h^{inv}[n] \), must satisfy

\[
h^{inv}[n] \ast h[n] = \delta[n]
\]

\[
H^{inv}(e^{j\Omega}) H(e^{j\Omega}) = 1
\]

The impulse response of the inverse system is then obtained as

\[
H^{inv}(e^{j\Omega}) = \frac{1}{H(e^{j\Omega})}
\]
3.10.3 Convolution of Periodic Signals

Basic Concept:

1. Define the periodic convolution of two CT signals \( x(t) \) and \( z(t) \), each having period \( T \), as

\[
y(t) = x(t) \ast z(t) = \int_{0}^{T} x(\tau) z(t - \tau) \, d\tau
\]

where the symbol \( \ast \) denotes that integration is performed over a single period of the signals involved.

\[
y(t) = x(t) \ast z(t) \quad \overset{FS; \frac{2\pi}{T}}{\leftrightarrow} \quad Y[k] = TX[k]Z[k] \quad (3.44)
\]

Convolution in Time-Domain \( \leftrightarrow \) Multiplication in Frequency-Domain
Example 3.36 Convolution of Two Periodic Signals

Evaluate the periodic convolution of the sinusoidal signal

\[ z(t) = 2 \cos(2\pi t) + \sin(4\pi t) \]

with the periodic square wave \( x(t) \) depicted in Fig. 3.56.

<Sol.>

1. Both \( x(t) \) and \( z(t) \) have fundamental period \( T = 1 \).

2. \( y(t) = x(t) \ast z(t) \)

\[ y(t) = x(t) \ast z(t) \quad \xrightarrow{FS; \frac{2\pi}{T}} \quad Y[k] = TX[k]Z[k] \]

3. Coefficients of FS representation of \( z(t) \):

\[
Z[k] = \begin{cases} 
1, & k = \pm 1 \\
\frac{1}{2j}, & k = 2 \\
\frac{-1}{2j}, & k = -2 \\
0, & \text{otherwise}
\end{cases}
\]
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Coefficients of FS representation of \( x(t) \):

\[
X[k] = \frac{2 \sin (k \pi /2)}{2k \pi}
\]

Example 3.13

3. FS Coefficients of \( y(t) \):

\[
Y[k] = X[k]Z[k] = \begin{cases} 1 / \pi & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}
\]

\[
y(t) = \left( \frac{2}{\pi} \right) \cos (2 \pi t)
\]

Define the discrete-time convolution of two \( N \)-periodic sequences \( x[n] \) and \( z[n] \) as

\[
y[n] = x[n] \otimes z[n] = \sum_{k=0}^{N-1} x[k]z[n-k]
\]

\[
y[n] = x[n] \otimes z[n] \quad \overset{\text{DTFS; } 2\pi}{\longrightarrow} \quad Y[k] = NX[k]Z[k]
\]

(3.46)

Convolution in Time-Domain \( \leftrightarrow \) Multiplication in Frequency-Domain
The convolution properties of all four Fourier representation are summarized in Table 3.5.

Table 3.5 Convolution Properties

<table>
<thead>
<tr>
<th>Operation</th>
<th>Continuous Domain</th>
<th>Discrete Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(t) \ast z(t)$</td>
<td>$X(j\omega)Z(j\omega)$</td>
<td>$TX[k]Z[k]$</td>
</tr>
<tr>
<td>$x(t) \bigodot z(t)$</td>
<td>$X(e^{j\Omega})Z(e^{j\Omega})$</td>
<td>$NX[k]Z[k]$</td>
</tr>
</tbody>
</table>

3.11 Differentiation and Integration Properties

- Differentiation and integration are operations that apply to continuous functions.

3.11.1 Differentiation in Time

1. A nonperiodic signal $x(t)$ and its FT, $X(j\omega)$, is related by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} \, d\omega$$

Differentiating both sides with respect to $t$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)j\omega e^{j\omega t} \, d\omega$$
Differentiation of $x(t)$ in Time-Domain $\leftrightarrow (j\omega) \times X(j\omega)$ in Frequency-Domain

\[ \mathcal{F}\left[ \frac{d}{dt} x(t) \right] \bigg|_{\omega=0} = (j\omega) \times X(j\omega) \bigg|_{\omega=0} = 0 \]

Example 3.37 Verifying the Differentiation Property

The differentiation property implies that

\[ \frac{d}{dt} \left( e^{-at} u(t) \right) \quad \leftrightarrow \quad \text{FT} \quad \frac{j\omega}{a + j\omega} \]

Verify this result by differentiating and taking the FT of the result.

<Sol.> 1. Using the product rule for differentiation, we have

\[ \frac{d}{dt} \left( e^{-at} u(t) \right) = -ae^{-at} u(t) + e^{-at} \delta(t) = -ae^{-at} u(t) + \delta(t) \]

2. Taking the FT of each term and using linearity, we may write

\[ \frac{d}{dt} \left( e^{-at} u(t) \right) \quad \leftrightarrow \quad \text{FT} \quad \frac{-a}{a + j\omega} + 1 \quad \leftrightarrow \quad \text{FT} \quad \frac{-j\omega}{a + j\omega} \]

Frequency response of a continuous system

1. System equation in terms of differential equation:
Fourier Representations of Signals & LTI Systems

1. The FS representation of a periodic signal $x(t)$:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j k \omega_0 t}$$

Differentiating both sides with respect to $t$:

$$\frac{d}{dt} x(t) = \sum_{k=-\infty}^{\infty} X[k] j k \omega_0 e^{j k \omega_0 t}$$

(3.47)

2. Differentiation property of a periodic signal:

$$\frac{d}{dt} x(t) \xrightarrow{FS; \omega_0} j k \omega_0 X[k]$$

2. Frequency response of the system:

Taking FT of both sides:

$$Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k$$

$$X(j\omega) = \sum_{k=0}^{N} a_k (j\omega)^k$$

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

The Differentiation property for a periodic signal

1. The FS representation of a periodic signal $x(t)$:

2. Differentiation property of a periodic signal:
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Differentiation of $x(t)$ in Time-Domain $\leftrightarrow (jk\omega_0) \times X[k]$ in Frequency-Domain

$$\mathcal{F}\left[\frac{d}{dt} x(t)\right] = (jk\omega_0) \times X[k]_{k=0} = 0$$

Example 3.39 Differentiation Property

Use the differentiation property to find the FS representation of the triangular wave depicted in Fig. 3.59 (a).

<Sol.>

1. Define a waveform:

$$z(t) = \frac{d}{dt} y(t)$$

Figure 3.59 (p. 274)

Signals for Example 3.39. (a) Triangular wave $y(t)$. (b) The derivative of $y(t)$ is the square wave $z(t)$. 
2. Comparing $z(t)$ in **Fig. 3.59 (b)** and the square wave $x(t)$ with $T_0/T = \frac{1}{4}$ in Example 3.13, we have

$$z(t) = 4x(t) - 2$$

$$Z[k] = 4X[k] - 2\delta[k]$$

3. The differentiation property implies that

$$Z[k] = jk\omega_0 Y[y]$$

$$Y[y] = \frac{1}{jk\omega_0}Z[k]$$

The quantity $Y[0]$ is the average value of $y(t)$ and is determined by inspection **Fig. 3.59 (b)** to be $T/2$. Therefore,

$$y(t) \xrightarrow{FS: \omega_0} Y[k] = \begin{cases} 
T/2, & k = 0 \\
2T \sin\left(\frac{k\pi}{2}\right), & k \neq 0 
\end{cases}$$
3.11.2 Differentiation in Frequency

1. FT of signal \( x(t) \):
\[
X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt
\]

Differentiating both sides with respect to \( \omega \):
\[
\frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{\infty} jtx(t)e^{-j\omega t} \, dt
\]

2. Differentiation property in frequency domain:
\[
- jtx(t) \leftrightarrow \frac{d}{d\omega} X(j\omega)
\]

\[\text{Example 3.40 FT of a Gaussian Pulse}\]

Use the differentiation-in-time and differentiation-in-frequency properties to determine the FT of the Gaussian pulse, depicted in Fig. 3.60 and defined by
\[
g(t) = \left( \frac{1}{\sqrt{2\pi}} \right) e^{-t^2/2}
\]

<Sol.>
1. The derivative of \( g(t) \) with respect to \( t \):
\[
\frac{d}{dt} g(t) = \left( \frac{-t}{\sqrt{2\pi}} \right) e^{-t^2/2} = -tg(t)
\]
2. Differentiation-in-time property:

\[ \frac{d}{dt} g(t) \xrightarrow{FT} j\omega G(j\omega) \]

\( (3.48) \)

\[ -t \cdot g(t) \xrightarrow{FT} j\omega G(j\omega) \]

\( (3.49) \)

3. Differentiation-in-time property:

\[ -t \cdot g(t) \xrightarrow{FT} \frac{1}{j} \frac{d}{d\omega} G(j\omega) \]

\( (3.50) \)

Since the left-hand sides of Eqs. (3.49) and (3.50) are equal, then we have

\[ \frac{d}{d\omega} G(j\omega) = -\omega G(j\omega) \]

Differential Equation of \( G(j\omega) \)

\[ G(j\omega) = ce^{-\omega^2/2} \]

4. The integration constant \( c \) is determined by noting that (see Appendix A-4)

\[ G(j0) = \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \right) e^{-t^2/2} dt = 1 \]

The FT of a Gaussian pulse is also a Gaussian pulse!

\[ \left( \frac{1}{\sqrt{2\pi}} \right) e^{-t^2/2} \xrightarrow{FT} e^{-\omega^2/2} \]
3.11.3 Integration

♀ The operation of integration applies only to continuous dependent variables.

In both FT and FS, we may integrate with respect to time.

In both FT and DTFT, we may integrate with respect to frequency.

♀ We limit our consideration to integrating nonperiodic signals

1. Define

\[ y(t) = \int_{-\infty}^{t} x(\tau) d\tau \]

\[ \frac{d}{dt} y(t) = x(t) \quad (3.51) \]

2. By differentiation property, we have

\[ Y(j\omega) = \frac{1}{j\omega} X(j\omega) \]

\[ (3.52) \]

Indeterminate at \( \omega = 0 \)

\[ X(j0) = 0 \]

3. When the average value of \( x(t) \) is not zero, then it is possible that \( y(t) \) is not square integrable.

FT of \( y(t) \) may not converge!

W can get around this problem by including impulse in the transform, i.e.

\[ Y(j\omega) = \frac{1}{j\omega} X(j\omega) + c\delta(\omega) \]

\[ c \text{ can be determined by the average value of } x(t) \]
4. General form of integration property:

\[
\int_{-\infty}^{t} x(\tau) d\tau \leftrightarrow \text{FT} \rightarrow \frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega) \quad (3.53)
\]

Ex. Demonstration for the integration property by deriving the FT of the unit step

1. Unit step signal:

\[
u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau
\]

2. FT of \( \delta(t) \):

\[
\delta(t) \leftrightarrow \text{FT} \rightarrow 1
\]

3. Eq. (3.53) suggests that:

\[
u(t) \leftrightarrow \text{FT} \rightarrow U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)
\]

♦ Check:

1. Unit step:

\[
u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t) \quad (3.54)
\]

2. Signum function:

\[
\text{sgn}(t) = \begin{cases} 
-1, & t < 0 \\
0, & t = 0 \\
1, & t > 0
\end{cases}
\]
3. Using the results of Example 3.28:

The transform of \( \text{sgn}(t) \) may be derived using the differentiation property because it has a zero time-average value. Let

\[
\frac{1}{2} \xrightarrow{FT} \pi \delta(\omega)
\]

We know that \( S(j0) = 0 \). This knowledge removes the indeterminacy at \( \omega = 0 \) associated with differentiation property, and we conclude that
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This relationship can be written as $S(j\omega) = \frac{2}{j\omega}$ with the understanding that $S(j0) = 0$.

4. Applying the linearity property to Eq. (3.54) gives the FT of $u(t)$:

$$u(t) \xrightarrow{FT} U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

This agrees exactly with the transform of the unit step obtained by using the integration property.

Table 3.6 Commonly Used Differentiation and Integration Properties.

The differentiation and integration properties of Fourier representation are summarized in Table 3.6.

<table>
<thead>
<tr>
<th>Property</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d}{dt} x(t)$</td>
<td>$j\omega X(j\omega)$</td>
</tr>
<tr>
<td>$\frac{d}{dt} x(t)$</td>
<td>$jk\omega_0 X[k]$</td>
</tr>
<tr>
<td>$- jtx(t)$</td>
<td>$\frac{d}{d\omega} X(j\omega)$</td>
</tr>
<tr>
<td>$- jnx[n]$</td>
<td>$\frac{d}{d\Omega} X(e^{j\Omega})$</td>
</tr>
<tr>
<td>$\int_{-\infty}^{t} x(\tau) d\tau$</td>
<td>$\frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega)$</td>
</tr>
</tbody>
</table>
3.12 Time- and Frequency-Shift Properties

3.12.1 Time-Shift Property

1. Let \( z(t) = x(t - t_0) \) be a time-shifted version of \( x(t) \).

2. FT of \( z(t) \):

\[
Z(j\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} x(t - t_0)e^{-j\omega t} \, dt
\]

3. Change variable by \( \tau = t - t_0 \):

\[
Z(j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_0)} \, d\tau = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)e^{-j\omega \tau} \, d\tau = e^{-j\omega t_0} X(j\omega)
\]

◆ Time-shifting of \( x(t) \) by \( t_0 \) in Time-Domain

\( \leftrightarrow (e^{-j\omega t_0}) \times X(j\omega) \) in Frequency-Domain

\[
|Z(j\omega)| = |X(j\omega)| \quad \text{and} \quad \arg\{Z(j\omega)\} = \arg\{X(j\omega)\} - \omega_0 t
\]

◆ The time-shifting properties of four Fourier representation are summarized in Table 3.7.
Table 3.7 Time-Shift Properties of Fourier Representations

\[
\begin{align*}
  x(t - t_0) & \xrightarrow{FT} e^{-j\omega_0 t_0} X(j\omega) \\
  x(t - t_0) & \xrightarrow{FS} \omega_0 e^{-jk \omega_0 t_0} X(k) \\
  x[n - n_0] & \xrightarrow{DTFT} e^{-j\Omega n_0} X(e^{j\Omega}) \\
  x[n - n_0] & \xrightarrow{DTFS; \Omega_0} e^{-jk \Omega_0 n_0} X[k]
\end{align*}
\]

Example 3.41 Finding an FT Using the Time-Shift Property

Use the FT of the rectangular pulse \( x(t) \) depicted in Fig. 3.62 (a) to determine the FT of the time-shift rectangular pulse \( z(t) \) depicted in Fig. 3.62 (b).

<Sol.>
1. Note that \( z(t) = x(t - T_1) \).
2. Time-shift property:
   \[
   Z(j\omega) = e^{-j\omega T_1} X(j\omega)
   \]
3. In Example 3.25, we obtained
   \[
   X(j\omega) = \frac{2}{\omega} \sin(\omega T_0)
   \]
   and
   \[
   Z(j\omega) = e^{-j\omega T_1} \frac{2}{\omega} \sin(\omega T_0)
   \]
Frequency response of a discrete-time system

1. Difference equation:

\[ \sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k] \]

Taking DTFT of both sides of this equation

\[ z[n - k] \overset{\text{DTFT}}{\longleftrightarrow} e^{-jk\Omega} Z(e^{j\Omega}) \]

\[ \sum_{k=0}^{N} a_k \left(e^{-j\Omega}\right)^k Y(e^{j\Omega}) = \sum_{k=0}^{M} b_k \left(e^{-j\Omega}\right)^k X(e^{j\Omega}) \]

\[ \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{\sum_{k=0}^{M} b_k \left(e^{-j\Omega}\right)^k}{\sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k} \]

2. Frequency response:

\[ H\left(e^{j\Omega}\right) = \frac{\sum_{k=0}^{M} b_k \left(e^{-j\Omega}\right)^k}{\sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k} \] (3.55)
3.12.2 **Frequency-Shift Property**

1. Suppose that:

\[ x(t) \xrightarrow{FT} X(j\omega) \quad \text{and} \quad z(t) \xrightarrow{FT} Z(j\omega) \]

2. By the definition of the inverse FT, we have

\[
z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(j\omega)e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j(\omega - \gamma))e^{j\omega t} d\omega
\]

Substituting variables \( \eta = \omega - \gamma \) into above Eq. gives

\[
z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\eta)e^{j(\eta+\gamma)t} d\eta = e^{j\gamma} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\eta)e^{j\eta t} d\eta = e^{j\gamma} x(t)
\]

◆ **Frequency-shifting of** \( X(j\omega) \) **by** \( \gamma \) **in Frequency-Domain** [i.e. \( X(j(\omega - \gamma)) \)]

\( \leftrightarrow (e^{j\gamma t}) \times x(t) \) **in Time-Domain**
The frequency-shift properties of Fourier representations are summarized in **Table 3.8**.

**Table 3.8 Frequency-Shift Properties of Fourier Representations**

<table>
<thead>
<tr>
<th>Property</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{jyt}x(t)$</td>
<td>$\xrightarrow{FT} X(j(\omega - \gamma))$</td>
</tr>
<tr>
<td>$e^{jk_0\omega_0 t}x(t)$</td>
<td>$\xrightarrow{FS; \omega_0} x[k - k_0]$</td>
</tr>
<tr>
<td>$e^{j\Gamma n}x[n]$</td>
<td>$\xrightarrow{DTFT} X[e^{j(\Omega - \Gamma)}]$</td>
</tr>
<tr>
<td>$e^{jk_0\Omega_0 n}x[n]$</td>
<td>$\xrightarrow{FS; \Omega_0} X[k - k_0]$</td>
</tr>
</tbody>
</table>

**Example 3.42 Finding an FT by Using the Frequency-Shift Property**

Use the frequency-shift property to determine the FT of the complex sinusoidal pulse:

$$z(t) = \begin{cases} e^{j10t}, & |t| < \pi \\ 0, & |t| > \pi \end{cases}$$

**<Sol.>**

1. Express $z(t)$ as the product of a complex sinusoid $e^{j10t}$ and a rectangular pulse
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2. Use the results of Example 3.25, we have

\[ x(t) \xrightarrow{FT} X(j\omega) = \frac{2}{\omega} \sin(\omega \pi) \]

Frequency-shift property

3. FT of \( z(t) \):

\[ e^{j10t}x(t) \xrightarrow{FT} X(j(\omega - 10)) \]

\[ z(t) \xrightarrow{FT} \frac{2}{\omega - 10} \sin((\omega - 10)\pi) \]

Example 3.43 Using multiple Properties to Find an FT

Find the FT of the signal

\[ x(t) = \frac{d}{dt} \{ (e^{-3t}u(t)) * (e^{-t}u(t-2)) \} \]

1. To solve this problem, it needs to use three properties: differentiation in time, convolution, and time shifting.
2. Let \( w(t) = e^{-3t}u(t) \) and \( v(t) = e^{-t}u(t-2) \)

\[
x(t) = \frac{d}{dt} \{ w(t) * v(t) \}
\]

Convolution and differentiation property

\[
X(j\omega) = j\omega \{ W(j\omega)V(j\omega) \}
\]

3. From the transform pair:

\[
e^{-at}u(t) \xleftarrow{FT} \frac{1}{a + j\omega}
\]

\[
W(j\omega) = \frac{1}{3 + j\omega}
\]

4. Use the same transform pair and the time-shift property to find \( V(j\omega) \) by first writing

\[
v(t) = e^{-2}e^{-(t-2)}u(t-2)
\]

\[
V(j\omega) = e^{-2} \frac{e^{-j2\omega}}{1 + j\omega}
\]

5. FT of \( x(t) \):

\[
X(j\omega) = e^{-2} \frac{j\omega e^{-j2\omega}}{(1 + j\omega)(3 + j\omega)}
\]
3.13 Finding Inverse Fourier Transforms by Using Partial-Fraction Expansions

3.13.1 Inverse Fourier Transform

Example 3.44

Find the impulse response for the following frequency response with $\omega_n = 10,000$ rads/s and $Q = 2/5$

Frequency response:

$$H(j\omega) = \frac{1}{(j\omega)^2 + \frac{\omega_n}{Q}(j\omega) + \omega_n^2}$$

<Sol.>

1. Frequency response:

$$H(j\omega) = \frac{1}{(j\omega)^2 + 25000(j\omega) + (10000)^2}$$

2. Partial-fraction expansion:

$$\frac{1}{(j\omega)^2 + 25000(j\omega) + (10000)^2} = \frac{-1/15000}{j\omega + 20000} + \frac{1/15000}{j\omega + 5000}$$
3. Impulse response:

\[ e^{\frac{dt}{u(t)}} \xrightarrow{FT} \frac{1}{j\omega - d} \]

\[ h(t) = \left(\frac{1}{15000}\right)\left(e^{-5000t} - e^{-20000t}\right)u(t) \]

### 3.13.2 Inverse Discrete-Time Fourier Transform

**Example 3.45 Inverse by Partial-Fraction Expansion**

Find the inverse DTFT of

\[ X(e^{j\Omega}) = \frac{-\frac{5}{6}e^{-j\Omega} + 5}{1 + \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}} \]

**<Sol.>**

1. Characteristic polynomial:

\[ v^2 + \frac{1}{6}v - \frac{1}{6} = 0 \]

2. The roots of above polynomial: \(d_1 = -\frac{1}{2}\) and \(d_2 = \frac{1}{3}\).

3. Partial-Fraction Expansion:

\[ \frac{-\frac{5}{6}e^{-j\Omega} + 5}{1 + \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}} = \frac{C_1}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{C_2}{1 - \frac{1}{3}e^{-j\Omega}} \]
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Coefficients $C_1$ and $C_2$

$$C_1 = \left(1 + \frac{1}{2} e^{-j\Omega}\right) \frac{-\frac{5}{6} e^{-j\Omega} + 5}{1 + \frac{1}{6} e^{-j\Omega} - \frac{1}{6} e^{-j2\Omega}}$$

$$C_2 = \left(1 - \frac{1}{3} e^{-j\Omega}\right) \frac{-\frac{5}{6} e^{-j\Omega} + 5}{1 + \frac{1}{6} e^{-j\Omega} - \frac{1}{6} e^{-j2\Omega}}$$

$$x[n] = 4\left(-\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$$

3.14 Multiplication Property

笙Non-periodic continuous-time signals

1. Non-periodic signals: $x(t)$, $z(t)$, and $y(t) = x(t)z(t)$. Find the FT of $y(t)$.

2. FT of $x(t)$ and $z(t)$:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\nu) e^{jv t} \, d\nu$$

and

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(j\eta) e^{j\eta t} \, d\eta$$
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\[ y(t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(jv)Z(j\eta)e^{j(\eta+v)t} \, d\eta \, dv \]

Change variable: \( \eta = \omega - v \)

\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jv)Z(j(\omega-v)) \, dv \right] e^{j\omega t} \, d\omega \]

3. FT of \( y(t) \):

\[ y(t) = x(t)z(t) \quad \xrightarrow{FT} \quad Y(j\omega) = \frac{1}{2\pi} X(j\omega) \ast Z(j\omega) \]

where

\[ X(j\omega) \ast Z(j\omega) = \int_{-\infty}^{\infty} X(jv)Z(j(\omega-v)) \, dv \]

◆ Multiplication of two signals in Time-Domain

\[ \leftrightarrow \text{Convolution in Frequency-Domain} \times (1/2\pi) \]
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★ Non-periodic discrete-time signals

1. Non-periodic DT signals: \( x[n], z[n], \) and \( y[n] = x[n]z[n] \).

2. DTFT of \( y[n] \):

\[
y[n] = x[n]z[n] \quad \xrightarrow{DTFT} \quad Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \odot Z(e^{j\Omega})
\]

where the symbol \( \odot \) denotes periodic convolution.

Here, \( X(e^{j\Omega}) \) and \( X(e^{j\Omega}) \) are \( 2\pi \)-periodic, so we evaluate the convolution over a \( 2\pi \) interval:

\[
X(j\omega) \odot Z(j\omega) = \int_{-\pi}^{\pi} X(e^{j\theta})Z(e^{j(\Omega-\theta)})d\theta
\]

◆ Multiplication of two signals in Time-Domain

\( \leftrightarrow \) Convolution in Frequency-Domain \( \times (1/2\pi) \)
Multiplication property can be used to study the effects of truncating a time-domain signal on its frequency-domain.

Windowing!

Truncate signal $x(t)$ by a window function $w(t)$ is represented by

$$y(t) = x(t)w(t)$$

Figure 3.65 (a)

Figure 3.65a (p. 293)
The effect of windowing.
(a) Truncating a signal in time by using a window function $w(t)$. 
1. FT of \( y(t) \):

\[
y(t) \xrightarrow{FT} Y(j\omega) = \frac{1}{2\pi} X(j\omega) \ast W(j\omega)
\]

2. If \( w(t) \) is the rectangular window depicted in Fig. 3.65 (b), we have

\[
W(j\omega) = \frac{2}{\omega} \sin(\omega T_0)
\]

Smoothing the details in \( X(j\omega) \) and introducing oscillation near discontinuities in \( X(j\omega) \).

**Figure 3.65b (p. 293)**
(b) Convolution of the signal and window FT’s resulting from truncation in time.
Multiplication of periodic time-domain signals corresponds to convolution of the Fourier representations.

\[
y(t) = x(t)z(t) \quad \overset{FS; 2\pi/T}{\longrightarrow} \quad Y[k] = X[k] * Z[k]
\]

(3.58)

where

\[
X[k] * Z[k] = \sum_{m=-\infty}^{\infty} X[m] Z[k-m]
\]

Non-periodic convolution of the FS coefficients

Same fundamental periods of \( x(t) \) and \( z(t) \)

Example 3.47 Radar Range Measurement: Spectrum of RF Pulse Train

The RF pulse train used to measure the range and introduced in Section 1.10 may be defined as the product of a square wave \( p(t) \) and a sine wave \( s(t) \), as shown in Fig. 3.68. Assume that \( s(t) = \sin \left( \frac{1000\pi t}{T} \right) \). Find the FS coefficients of \( x(t) \).

<Sol.>

1. Suppose that \( s(t) = p(t) x(t) \). Multiplication property

\[
X[k] = P[k] * S[k]
\]
The RF pulse is expressed as the product of a periodic square wave and a sine wave.

Figure 3.68 (p. 297)
2. Fundamental period:
For $p(t)$: $\omega_0 = 2\pi/T$.

Fundamental frequency

$s(t) = \sin\left(1000\pi t / T\right) = \sin(500\omega_0 t)$.

3. FS coefficients of $s(t)$:

$$S[k] = \begin{cases} 
1/(2j), & k = 500 \\
-1/(2j), & k = -500 \\
0, & \text{otherwise}
\end{cases}$$

4. FS coefficients of $p(t)$:

$$P[k] = e^{-jk\omega_o T_o} \frac{\sin(k\omega_o T_o)}{k\pi}$$

5. FS coefficients of $x(t)$:

$$X[k] = \frac{1}{2} e^{-j(k-500)\omega_o T_o} \frac{\sin((k - 500)\omega_o T_o)}{(k - 500)\pi} - \frac{1}{2} e^{-j(k+500)\omega_o T_o} \frac{\sin((k + 500)\omega_o T_o)}{(k + 500)\pi}.$$
Figure 3.69 (p. 298)
FS magnitude spectrum for $0 \leq k \leq 1000$. The result is depicted as a continuous curve, due to the difficulty of displaying 1000 stems.
Multiplication property for discrete-time periodic signals:

\[ y[n] = x[n]z[n] \quad \leftrightarrow \quad DTFS: \frac{2\pi}{T} \quad Y[k] = X[k] \otimes Z[k] \quad (3.59) \]

where

\[ X[k] \otimes Z[k] = \sum_{m=0}^{N-1} X[m] Z[k-m] \]

◆ Fundamental period = N.

◆ The multiplication properties of all four Fourier representations are summarized in Table 3.9.

Table 3.9 Multiplication Properties of Fourier Representations

<table>
<thead>
<tr>
<th>Representation</th>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>FT</td>
<td>(x(t)z(t))</td>
<td>(\frac{1}{2\pi} X(j\omega) * Z(j\omega))</td>
</tr>
<tr>
<td>FS</td>
<td>(x(t)z(t))</td>
<td>(X[k] * Z[k])</td>
</tr>
<tr>
<td>DTFS</td>
<td>(x[n]z[n])</td>
<td>(\frac{1}{2\pi} X(e^{j\Omega}) \otimes Z(e^{j\Omega}))</td>
</tr>
<tr>
<td>DTFT</td>
<td>(x[n]z[n])</td>
<td>(X[k] \otimes Z[k])</td>
</tr>
</tbody>
</table>
3.15 Scaling Property

1. Let \( z(t) = x(at) \).

2. FT of \( z(t) \):

\[
Z(j\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt
\]

Changing variable: \( \tau = at \)

\[
Z(j\omega) = \begin{cases} 
(1/a) \int_{-\infty}^{\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a > 0 \\
(1/a) \int_{-\infty}^{\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a < 0 
\end{cases}
\]

\[
z(t) = x(at) \xrightarrow{FT} (1/|a|) X(j\omega/a).
\]

**Scaling in Time-Domain \( \leftrightarrow \) Inverse Scaling in Frequency-Domain**

**Signal expansion or compression!**

Refer to **Fig. 3.70.**
Figure 3.70 (p. 300)
The FT scaling property. The figure assumes that $0 < a < 1$. 
$V \equiv \omega$
Example 3.48 *Scaling a Rectangular Pulse*

Let the rectangular pulse 

\[ x(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases} \]

Use the FT of \( x(t) \) and the scaling property to find the FT of the scaled rectangular pulse 

\[ y(t) = \begin{cases} 1, & |t| < 2 \\ 0, & |t| > 2 \end{cases} \]

**<Sol.>**

1. Substituting \( T_0 = 1 \) into the result of Example 3.25 gives 

\[ X(j\omega) = \frac{2}{\omega} \sin(\omega) \]

2. Note that \( y(t) = x(t/2) \).

\[ Y(j\omega) = 2X(j2\omega) = 2 \left( \frac{2}{2\omega} \right) \sin(2\omega) = \frac{2}{\omega} \sin(2\omega) \]

*Scaling property and \( a = 1/2 \)*

*Substituting \( T_0 = 2 \) into the result of Example 3.25 can also give the answer!*

**Example 3.49 Using Multiple Performance to Find an Inverse FT**

Find \( x(t) \) if 

\[ X(j\omega) = j \frac{d}{d\omega} \left\{ \frac{e^{i2\omega}}{1 + j(\omega / 3)} \right\} \]
Figure 3.71 (p. 301)
Application of the FT scaling property in Example 3.48. (a) Original time signal. (b) Original FT. (c) Scaled time signal \( y(t) = x(t/2) \). (d) Scaled FT \( Y(j\omega) = 2X(j2\omega) \).
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<Sol.>

1. Differentiation in frequency, time shifting, and scaling property to be used to solve the problem.

2. Transform pair:

\[ s(t) = e^{-t}u(t) \quad \overset{FT}{\rightarrow} \quad S(j\omega) = \frac{1}{1 + j\omega} \]

\[ X(j\omega) = j \frac{d}{d\omega} \{ e^{j2\omega} S(j\omega/3) \} \]

3. Treatment procedure: scaling → time-shift → differentiation

Define \( Y(j\omega) = S(j\omega/3) \).

\[ y(t) = 3s(3t) = 3e^{-3t}u(3t) = 3e^{-3t}u(t) \]

Define \( W(j\omega) = e^{j2\omega} Y(j\omega/3) \).

\[ w(t) = y(t + 2) = 3e^{-3(t+2)}u(t + 2) \]

Since

\[ X(j\omega) = j \frac{d}{d\omega} W(j\omega) \]

\[ x(t) = tw(t) = 3te^{-3(t+2)}u(t + 2) \]
Scaling property in FS:

1. If \( x(t) \) is a periodic signal, then \( z(t) = x(at) \) is also periodic.
2. For convenience, we assume \( a > 0 \).
3. If \( x(t) \) has fundamental period \( T \), then \( z(t) \) has fundamental period \( T/a \).
4. If the fundamental frequency of \( x(t) \) is \( \omega_0 \), then the fundamental frequency of \( z(t) \) is \( a\omega_0 \).
5. FS coefficients of \( z(t) \):

\[
Z[k] = \frac{a}{T} \int_{0}^{T/a} z(t)e^{-jk\omega_0 t} dt
\]

6. FS coefficients of \( x(t) \) and \( x(at) \) are identical, only the harmonic spacing changes from \( \omega_0 \) to \( a\omega_0 \).

Scaling property in Discrete-Time Domain:

\( z[n] = x[\rho n] \)

1. First of all, \( z[n] = x[\rho n] \) is defined only for integer values of \( \rho \).
2. If \( |\rho| > 1 \), then scaling operation discards information.
3.16 Parseval Relationships

♦ The energy or power in the time-domain representation of a signal is equal to the energy or power in the frequency-domain representation.

◆ Case for CT nonperiodic signal: \( x(t) \)

1. Energy in \( x(t) \):

\[
W_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt
\]

2. Note that

\[
| x(t) |^2 = x(t) x^*(t)
\]

Express \( x^*(t) \) in terms of its FT \( X(j\omega) \):

\[
x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t} \, d\omega
\]

The integral inside the braces is the FT of \( x(t) \).
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\[ \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega \]  

(3.62)

◆ Energy in Time-Domain Representation
\[ \leftrightarrow \] Energy in Frequency-Domain Representation \( \times (1/2\pi) \)

◆ The Parseval Relationships of all four Fourier representations are summarized in Table 3.10.

**Table 3.10 Parseval Relationships for the Four Fourier Representations**

<table>
<thead>
<tr>
<th>Representations</th>
<th>Parseval Relationships</th>
</tr>
</thead>
<tbody>
<tr>
<td>FT</td>
<td>[ \int_{-\infty}^{\infty}</td>
</tr>
<tr>
<td>FS</td>
<td>[ \frac{1}{T} \int_{0}^{T}</td>
</tr>
<tr>
<td>DTFT</td>
<td>[ \sum_{n=-\infty}^{\infty}</td>
</tr>
<tr>
<td>DTFS</td>
<td>[ \frac{1}{N} \sum_{n=0}^{N-1}</td>
</tr>
</tbody>
</table>
Example 3.50 Calculating Energy in a Signal

Let

\[ x[n] = \frac{\sin(Wn)}{\pi n} \]

Use Parseval’s theorem to evaluate

\[ \chi = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} \frac{\sin^2(Wn)}{\pi^2 n^2} \]

Direct calculation in time-domain is very difficult!

<Sol.>

1. Using the DTFT Parseval relationship in Table 3.10, we have

\[ \chi = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega \]

2. Since

\[ x[n] \xrightarrow{DTFT} X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| \leq W \\ 0, & W < |\Omega| < \pi \end{cases} \]

\[ \chi = \frac{1}{2\pi} \int_{-W}^{W} 1 d\Omega = \frac{W}{\pi} \]
3.18 Duality

**The Duality Property of the FT**

1. FT pair:

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} \, d\omega \quad \text{and} \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt
\]

2. General equation:

\[
y(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(\eta)e^{j\nu \eta} \, d\nu
\]  

\[(3.66)\]

Choose \( \nu = t \) and \( \eta = \omega \) the Eq. (3.66) implies that

Rectangular in time-domain \( \leftrightarrow \) sinc function in frequency-domain

Figure 3.73 (p. 307)
Duality of rectangular pulses and sinc functions.
Fourier Representations of Signals & LTI Systems

3. Interchange the role of time and frequency by letting $\nu = -\omega$ and $\eta = t$, then Eq. (3.66) implies that

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(\omega) e^{j\omega t} d\omega$$

(3.67)

$$y(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

(3.68)

Example 3.52 Applying Duality

Find the FT of

$$x(t) = \frac{1}{1 + jt}$$

<Sol.>

$$f(t) = e^{-t}u(t) \quad \longleftrightarrow \quad F(j\omega) = \frac{1}{1 + j\omega}$$

Note: $t \rightarrow -\omega$

$\omega \rightarrow t$

$$F(jt) = \frac{1}{1 + jt} \quad \longleftrightarrow \quad 2\pi f(-\omega) = 2\pi e^{\omega} u(-\omega)$$
Figure 3.74 (p. 309)
The FT duality property.

\[ v \equiv \omega; \quad p \equiv \pi \]
### 3.18.2 The Duality Property of the DTFS

1. DTFS Pair for \( x[n] \):

\[
x[n] = \sum_{k=0}^{N-1} X[k] e^{j\Omega_0 n}
\]

and

\[
X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\Omega_0 n}
\]

2. The DTFS duality is stated as follows: if

\[
x[n] \quad \overset{\text{DTFS}}{\longleftrightarrow} \quad \frac{2\pi}{N} \quad X[k]
\]

(3.71)

\[
X[n] \quad \overset{\text{DTFS}}{\longleftrightarrow} \quad \frac{2\pi}{N} \quad \frac{1}{N} x[-k]
\]

(3.72)

### 3.18.3 The Duality Property of the DTFT and FS

1. FS of a periodic continuous time signal \( z(t) \):

\[
z(t) = \sum_{k=-\infty}^{\infty} Z[k] e^{j\omega_0 t} \quad \overset{\text{FS}}{\longleftrightarrow} \quad Z[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} z(t) e^{-jkt} dt
\]

2. DTFT of an nonperiodic discrete-time signal \( x[n] \):

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{jn\Omega} d\Omega \quad \overset{\text{DTFS}}{\longleftrightarrow} \quad X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}
\]
3. Duality relationship between $z(t)$ and $X(e^{j\Omega})$

Require $z(t)$ and $X(e^{j\Omega})$ has same period.

$T = 2\pi$

Assumption: $\omega_0 = 1$

$\Omega$ in the DTFT $\leftrightarrow t$ in the FS

$n$ in the DTFT $\leftrightarrow -k$ in the FS

4. Duality property between the FS and the DTFT: if

$$
x[n] \overset{\text{DTFS}}{\rightarrow} X(e^{j\Omega})$$

(3.73)

$$
X(e^{jt}) \overset{\text{FS;1}}{\rightarrow} x[-k]
$$

(3.74)

◆ The Duality Properties of Fourier representations are summarized in Table 3.11

**Table 3.11 Duality Properties of Fourier Representations**

<table>
<thead>
<tr>
<th></th>
<th>FT $f(t)$</th>
<th>$\overset{\text{FT}}{\rightarrow}$</th>
<th>$F(j\omega)$</th>
<th>$\overset{\text{FT}}{\rightarrow}$</th>
<th>$2\pi f(-\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DTFS</td>
<td>$x[n]$</td>
<td>$\overset{\text{DTFS;2\pi/N}}{\rightarrow}$</td>
<td>$X[k]$</td>
<td>$\overset{\text{DTFS;2\pi/N}}{\rightarrow}$</td>
<td>$(1/N)x[-k]$</td>
</tr>
<tr>
<td>FS-DTFT</td>
<td>$x[n]$</td>
<td>$\overset{\text{DTFS}}{\rightarrow}$</td>
<td>$X(e^{j\Omega})$</td>
<td>$\overset{\text{FS;1}}{\rightarrow}$</td>
<td>$x[-k]$</td>
</tr>
</tbody>
</table>